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BIOGRAPHY.

HOÜEL.

BY GEORGE BRUCE HALSTED.

GUILLAUME JULES HOÜEL, of a very old protestant family of Normandy, was born at Thaon (Calvados) on April 7th, 1823, and died at Périers, near Caen, June 14th, 1886.

The key to his whole mental life was this old protestant blood, which means so much in a Roman catholic country.

After studying at the lyceum of Caen, and the college Rollin, he entered the great Normal School of Paris in 1843. On leaving, he taught at Bourges, Bordeaux, Pau, Alençon and Caen.

In 1855 he took his Doctor's degree at the Sorbonne, and then, declining the overtures of Le Verrier to join the working force at the Observatory, he retired to his home at Thaon to continue his researches. In 1859 he was called to succeed Le Besgue in the chair of pure mathematics of the Faculty of Sciences of Bordeaux. Here he found dignity and facilities for work, and considered the position as final.

The idea of duty was the essence of his character, remarkably sweet and even. Not only in his official position did he forward science, he spread it with profusion all around him.

So precise and rigorous was his mind that he scorned Legendre's and Clairant's geometries, and the conventional neatness of the French texts, in favor of the eternal geometer Euclid.

In 1863 he published at Greifswald an essay on the fundamental principles of geometry. In this he has already reached by himself the idea that a demonstration of the postulatum of Euclid is impossible. He says: "Since long, the scientific researches of mathematicians on the fundamental principles of elementary geometry have concentrated themselves almost exclusively on the theory of parallels; and if, hitherto, the efforts of so many eminent minds have produced no satisfactory result, it is perhaps permitted to conclude thence that in pursuing these researches they have followed a false path and attacked an insoluble problem, of which the importance has been exaggerated in consequence of inexact ideas on the nature and origin of the primordial verities of the science of space."

To the mind so self-prepared came an important communication in 1866 from Dr. R. Baltzer informing Hoüel of the fundamental idea of Lobachévski and Bolyai and announcing that Baltzer would mention it in the forthcoming second edition of his Elements of Geometry. That very year 1866 Hoüel issued his translation of Lobachévski's "Geometrische Untersuchungen zur Theorie der Parallellinien," and in the preface to his translation quotes from W. Bolyai's "Kurzer Grundriss eines Versuchs etc.," and mentions the work of J. Bolyai with date 1832. In this preface he says: "The aim of the author is to prove that there exists *à priori* no reason to affirm that the sum of the three angles of a rectilineal triangle is not less than two right angles, or, what comes to the same thing, that one cannot draw, through a given point more than a single straight not meeting a given straight in the same plane.

In spite of the high value of these researches, they have not hitherto drawn the attention of any geometer. We do not believe however that we exaggerate their philosophic import in saying that they throw a new day on the fundamental principles of geometry, and that they open a path yet unexplored capable of leading to unexpected discoveries. Not to go beyond elementary questions, one cannot deny that they accomplish an immense advance in methods of teaching by relegating among the chimeras the hope still nourished by so many geometers of demonstrating the postulatum of Euclid.

Henceforth these attempts must be ranked with the quadrature of the circle and perpetual motion."

He mentions the assumption, (three points are costraight or concyclic), given by W. Bolyai to replace Euclid's.

A translation of J. Bolyai was delayed until 1868 by Hoüel's inability to procure a copy of the now celebrated Appendix. How this difficulty was fortunately overcome I learned while in Hungary where my friend Franz Schmidt entrusted to me a precious file of Hoüel's own letters. From these letters it appears that a copy of Hoüel's 'Essai' of 1863 having come by chance into the hands of a young architect of Temesvár in Hungary, this youth (Franz Schmidt), desirous of continuing his mathematical studies wrote for counsel to Hoüel. Hoüel had answered helpfully, and later implored the aid of Schmidt to procure Bolyai's work, and besought Schmidt to collect what materials he could for a biography. This Schmidt did, and his article on Grunert's Archiv, 1868,

remained, until my own researches and my journey to Hungary, the only source of information on these wonderful Magyars. Schmidt succeeded in procuring for Hoüel two copies of Bolyai's work. One Hoüel proceeded to translate himself; the other he sent to Battaglini, asking him to make known in Italy this wonderful idea. This he did by an Italian translation. Thus to Hoüel belongs a perfectly definite and permanent place in the final history of human thought.

Much else he did; so much that I could not attempt to enumerate it in the brief space at my disposal here. Fortunately it has been most sympathetically done by M. G. Brunel in a book of 78 pages most obligingly furnished me by Hoüel's son-in-law, Monsieur H. Barckhausen.

M. Brunel cites on page 34 my Bibliography of Hyper-Space and Non-Euclidean Geometry (1878), and also that published at Kiev in 1880 by Vashtchenko-Zaharchenko, but omits to state that this latter was simply a reprint of mine with slight additions, as is also that given at the end of the Kazan edition of Lobachévski's Works, 1886. Some grotesque effects are produced by reprinting or attempting to reprint my English. Thus under P. G. Tait the title of the work is given as follows: "Mentions Hyper-Space in his Address as Pres. of Math. Sect. of Brit. Assoc. at Edinbvrgh." Under G. P. Young we read "The relation which can be proved to subsist between the Area of a Plane Triangle and the Sum of the Hypothesis that Euclid's twelfth Axiom is false." I must take it, from this extraordinary summation of a hypothesis, that English is nearly as difficult as Russian, though neither can for one instant compete with the Magyar.

In his personal character Hoüel reached that perfection which he has done so much to introduce into the foundations of Geometry.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from March Number.]

PROPOSITION XXVI. *If the aforesaid AX, BX (fig. 31.) must indeed meet each other, but only at their infinite production toward the parts of the point X: I say there will be no assignable point T in AB, from which a perpendicular erected towards the parts of AX does not at a finite or terminated distance meet this AX in some point F.*

Demonstratur. For (from the preceding hypothesis) there will be in *AX*

some point N, from which the perpendicular NK let fall to BX is less than any assigned length, as suppose this TB.

But then is assumed in TB a portion CB equal to NK, and CN is joined. In the hypothesis of acute angle it is known that the angle NCB will be acute. Therefore (from Eu. I. 13) NCT, which is the adjacent angle, will be obtuse.

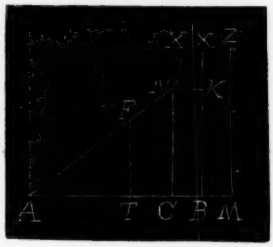


Fig. 31.

Therefore the straight which is erected toward the parts of AX perpendicularly from the point T (disposed between the points A and C), does not meet (from Eu. I. 17) CN at any point; and therefore (lest it should enclose a space with AT, or with TC) it strikes the terminated AN in some point F.

Therefore even in the hypothesis of acute angle (which we know can here alone hinder) there will be in this AB no assignable point T, from which the perpendicular erected toward the parts of AX does not, at a finite or terminated distance, meet this AX in a certain point F. Quod etc.

Corollary I. But thence follows, that, point M being assumed in AB produced, from which towards the parts of the point X is erected a perpendicular MZ, this cannot, even if infinitely produced, meet the aforesaid AX; because otherwise that other straight BX must (from the foregoing demonstration) at a finite distance meet this AX; which is against the present hypothesis.

Corollary II. From which again follows, that every perpendicular erected from any point, but not however infinitely removed, of this AB produced indefinitely, must at a finite distance meet the aforesaid AX, as soon as indeed it is assumed that *every* such perpendicular ever more, without any certain limit, approaches the other ever produced straight AX.

Corollary III. Whence finally follows, that not even at its infinite production can BX be cut by that AX; because otherwise from any point of that AX beyond the aforesaid intersection a certain perpendicular ZM could be supposed let fall to AB produced; whence again would follow, that BX (against the present hypothesis) met the aforesaid AX not at an infinite, but wholly at a finite distance.

But this last dictum is beyond necessity.

[Saccheri here handles a point at infinity, or *figurative* point, as if it were a *proper* point. Upon the extent to which he realized this to be unallowable, depends his real mental attitude toward the non-Euclidean geometries he had discovered. Did he intend his work to suggest what he would not have been allowed to print?]

A METHOD FOR DEVELOPING $\cos^n\theta$ AND $\sin^n\theta$.

By M. C. STEVENS, A. M., Department of Mathematics, Purdue University, Lafayette, Indiana.

De Morgan in his Calculus gives a method for expanding $\cos^n\theta$ and $\sin^n\theta$ when n is an integer which I have not noticed in any of our American works on that subject. As it leads to an easy method for integrating such expressions as

$$\int \cos^n\theta d\theta, \quad \int \sin^n\theta d\theta,$$

etc. I have thought it might be of interest to some of the readers of the MONTHLY. The method is as follows:

$\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$. Let $e^{i\theta}=x$, then $e^{-i\theta}=\dfrac{1}{x}$, and $\cos\theta=\frac{1}{2}\left(x+\dfrac{1}{x}\right)$.....(1)

$$e^{ni\theta}=x^n, \text{ then } e^{-ni\theta}=\frac{1}{x^n}, \cos n\theta=\tfrac{1}{2}\left(x^n+\frac{1}{x^n}\right).$$

Then from (1) $\cos^n\theta = \dfrac{1}{2^{n-1}}\left[\frac{1}{2}\left(x^n+\dfrac{1}{x^n}\right)+ n\frac{1}{2}\left(x^{n-2}+\dfrac{1}{x^{n-2}}\right)\right.$

$$\left. + \frac{n(n-1)}{2}\frac{1}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right).....\right.$$

$$= \frac{1}{2^{n-1}}\left[\cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2}\cos(n-4)\theta + ...\right.$$

If n be an even number$=2m$, there will be $2m+1$ terms in the development, which will give m cosines, namely, those of $2m\theta$, $2(m-1)\theta$.....down to 2θ, and an additional term which will not contain θ, the value of which is $\dfrac{2m(2m-1).....m+1}{\lfloor m}$. But if n be odd, and$=2m+1$, then there are $2m+2$ terms giving $m+1$ cosines, namely, those of $(2m+1)\theta$, $(2m-1)\theta$..... down to θ, with no middle term. Thus we have

$$\cos^6\theta = \tfrac{1}{3}\tfrac{1}{2}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$

Whence the integral of $\cos^6\theta d\theta = \frac{1}{1}\frac{1}{9}\frac{1}{2}\sin 6\theta + \frac{3}{6}\frac{3}{4}\sin 4\theta + \frac{1}{6}\frac{5}{4}\sin 2\theta + \frac{5}{1}\frac{5}{6}\theta$.

Also $\cos^7\theta = \frac{1}{6}\frac{1}{4}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$.

Whence $\int \cos^7\theta d\theta = \frac{1}{4}\frac{1}{4}\frac{1}{8}\sin 7\theta + \frac{7}{3}\frac{7}{2}\frac{7}{0}\sin 5\theta + \frac{7}{6}\frac{7}{4}\sin 3\theta + \frac{3}{6}\frac{5}{4}\sin\theta$.

The advantage of this method will be still more apparent by integrating* $\cos^3 3\theta\cos\theta d\theta$. Here $\cos^3 3\theta = \frac{1}{8}(x^3+x^{-3})^3 = \frac{1}{8}(x^9+x^{-9})+3(x^3+x^{-3})$.

Multiplying this by $\frac{1}{2}(x+x^{-1})$ we at once have

$$\cos^3 3\theta\cos\theta = \tfrac{1}{8}\cos 10\theta + \tfrac{1}{8}\cos 8\theta + \tfrac{3}{8}\cos 4\theta + \tfrac{3}{8}\cos 2\theta.$$

Whence $\int \cos^3 3\theta\cos\theta d\theta = \frac{1}{80}\sin 10\theta + \frac{1}{64}\sin 8\theta + \frac{3}{32}\sin 4\theta + \frac{3}{16}\sin 2\theta$.

It will be noticed that this form is well adapted for substituting values as limits of integration. For instance if the inferior limit be 0, and the superior limit $\frac{1}{6}\pi$ then $\frac{1}{80}\sin\frac{10}{6}\pi = -\frac{1}{160}\sqrt{3}$; $\frac{1}{64}\sin\frac{8}{6}\pi = -\frac{1}{128}\sqrt{3}$; $\frac{3}{32}\sin\frac{4}{6}\pi = \frac{3}{64}\sqrt{3}$; $\frac{3}{16}\sin 2\theta = \frac{3}{32}\sqrt{3}$.

$$\therefore \int_0^{\frac{1}{6}\pi} \cos^3 3\theta\cos\theta d\theta = \tfrac{81}{640}\sqrt{3}.$$

The reader will have no difficulty in applying the same method to develop $\sin^n\theta$ and then for integrating $\sin^n\theta d\theta$.

It will be observed that when we put $\cos\theta = \frac{1}{2}(x+\frac{1}{x})$ we do not escape the impossible; for this is as much an impossible form as $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$ for $x+\frac{1}{x}$ can never be *less* than 2, and $2\cos\theta$ can never be *greater* than 2.

CONCERNING CONICS THROUGH FOUR POINTS.

By EDGAR H. JOHNSON, Professor of Mathematics, Emory College, Oxford, Georgia.

The equation of the conic through a_1b_1, a_2b_2, a_3b_3, a_4b_4, and a fifth point x_1y_1 is

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1 & x_1 & y_1 & 1 \\ a_1^2 & a_1b_1 & b_1^2 & a_1 & b_1 & 1 \\ a_2^2 & a_2b_2 & b_2^2 & a_2 & b_2 & 1 \\ a_3^2 & a_3b_3 & b_3^2 & a_3 & b_3 & 1 \\ a_4^2 & a_4b_4 & b_4^2 & a_4 & b_4 & 1 \end{vmatrix} = 0,$$

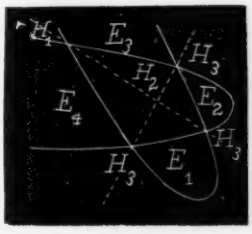
or $Ax^2+2Bxy+Cy^2+2Fx+2Gy+H=0$, where the coefficients A, B, C, ... are of the second degree in x_1 and y_1. The conic is an ellipse, parabola, or hy-

*Professor Waldo first called my attention to this easy method for integrating this particular expression.

perbola according as $AC-B^2$ is greater than, equal to, or less than zero. Through every point of the curve $AC-B^2=0$ (x_1 and y_1 being now general coordinates) may be drawn a parabola also passing through the four given points. Now it is known that through four points two parabolas can be drawn, the parabola being real or imaginary according as one of the four points does not or does lie in the triangle formed by the other three. (See Salmon's Conic Sections, page 153, ex. 1; or C. Smith's Conic Sections, pages 233-4).

Since through every point of each of these two parabolas, a parabola passing through the four given points is possible, the curve $AC-B^2=0$, of the fourth degree, decomposes into these same two parabolas.

Since $AC-B^2$ changes sign when a point crosses the curve, we have determined the locus of those points which with the four given points determine an ellipse (or hyperbola). The curve divides the plane into regions of two kinds, those for which $AC-B^2$ is positive, and those for which $AC-B^2$ is negative. Every point in a region of the first kind determines with the four given points an ellipse; every point of the second kind determines likewise an hyperbola. The points within the region enclosed by the two parabolas determine hyperbolas, since the four points determine a pair of straight lines, passing through this region, and for a pair of straight lines $AC-B^2<0$. Points in the regions marked H (see figure) determine with the four points of intersection of parabolas conics which are hyperbolas; points in the regions marked E determine likewise ellipses. A particular case of special interest arises when the four points become two pairs of coincident points, and the system becomes that of conics tangent to two given lines at given points. It is easy to show that the two parabolas become coincident. $AC-B^2$ is then a square and cannot change sign. The two tangents constitute one conic of the system and for the present purpose a pair of straight lines is a hyperbola. Hence all conics of the system, with the exceptions of the parabola and the pair of tangent lines, are hyperbolas.



In the above we have supposed points and conics to be real. It is easy to see that the condition for the passing of a real ellipse through four distinct real points is the same as for a real hyperbola. A real parabola can always be drawn through four real points not in the same straight line.

INTEGRAL SIDES OF RIGHT TRIANGLES.

By M. A. GRUBER, A. M., War Department, Washington, D. C.

$$a^2+b^2=c^2.$$

Problem I. To find integral sides of right triangles.

Rule 1. Take two integers, both odd or both even. ½ the sum of their squares equals the hypotenuse, or c; ½ the difference of their squares equals one of the legs, or b; and their product equals the other leg, or a.

Rule 2. Take any two integers. The sum of their squares equals the hypotenuse, or c; the difference of their squares equals one of the legs, or b; and twice their product equals the other leg, or a.

Rule 3. If *prime* integral sides are desired, the integers chosen must be prime to each other; in Rule 1, both odd; and in Rule 2, one odd and the other even.

Note. Rules 1 and 2 hold good also for fractional values. These rules are deduced from the two formulas mentioned in Problem II, and, to avoid repetition, are not discussed in this problem.

Problem II. Given one of the legs of a right triangle of integral sides to find the other leg and the hypotenuse.

The sides of a right triangle depend upon the equation $a^2+b^2=c^2$, in which a and b are the legs and c the hypotenuse of the triangle.

In the discussion of this problem, a is taken as the *given leg.*

When integral equations of the form $a^2+b^2=c^2$ are considered, the sets of values for a, b, and c are divided into two classes: (1) Those having no common factor; a, b, and c being prime integral values. (2) Those having a common factor; a, b, and c being found by multiplying a, b, and c of the first class by the highest common factor.

Sets of *prime* integral values are, therefore, the basis of work.

In right triangles of integral sides, any integer from 3 up may be taken as the value of one of the legs.

There are three kinds of integers to be considered: (1) Odd numbers; (2) Even numbers divisible by 4; and (3) Even numbers that are 2 times an odd number.

a may, then, be any one of these three kinds of numbers.

When a is *an odd number*, we have the formula

$$(mn)^2+\left(\frac{m^2-n^2}{2}\right)^2=\left(\frac{m^2+n^2}{2}\right)^2$$

by means of which to find b and c, so that a, b, and c have no common factor.

$$mn=a,\ \frac{m^2-n^2}{2}=b,\text{ and }\frac{m^2+n^2}{2}=c.$$

m and n are odd and are prime to each other, and $m>n$. There are as many

sets of prime integral values of a, b, and c as m and n can be made sets of odd, prime, integral factors, the product of each set of which factors equals a.

When a is *an even number divisible by* 4, we have the formula $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, by means of which to find b and c, so that a, b, and c have no common factor. $2mn=a$, $m^2-n^2=b$, and $m^2+n^2=c$. m and n are prime to each other, one being odd, the other even; and $m>n$. There are as many sets of prime integral values of a, b, and c as m and n can be made sets of prime integral factors, the product of each set of which factors equals $\frac{1}{4}a$.

When a is *an even number that is* 2 *times an odd number*, we first find the set or sets of values for a equal to the odd number, and then multiply them by 2.

When a contains odd factors other than itself and unity, or even factors divisible by 4, there are other sets of values, in which a, b, and c have a common factor. There are as many sets of values of this kind as the sets of prime integral values that can be found for the odd factors and the even factors divisible by 4, cantained in a. In this case we first find the sets of prime integral values for each of the factors and then multiply them by the respective numbers that produce a.

In problems relating to the integral sides of right triangles, unity and the number itself are considered factors of a number.

For the purpose of bringing out the foregoing statements more clearly to the mind of the reader, we shall present them by way of illustration.

Put $a=3$, the lowest integer for integral sides of right triangles. Then $mn=3=3\times1$; whence $m=3$, $n=1$. Substituting these values in the formula for a=an odd number, we find $b=\frac{1}{2}(3^2-1^2)=4$, and $c=\frac{1}{2}(3^2+1^2)=5$. There is but one set of values; viz., 3, 4, 5.

Put $a=4$. Then $2mn=4=2\times2\times1$; whence $m=2$, $n=1$. Substituting these values in the formula for a=an even number divisible by 4, we find $b=2^2-1^2=3$, and $c=2^2+1^2=5$. This set of values, 4, 3, 5, is the same as that for $a=3$, only a and b have interchanged values. There is but one set.

Put $a=12$. Then $2mn=12=2\times6\times1$ and $2\times3\times2$. There are, therefore, two sets of prime integral values. To find first set, $m=6$, $n=1$. To find second set, $m=3$, $n=2$. Whence the sets are 12, 35, 37; and 12, 5, 13. But $12=4\times3$ and 3×4. Hence there are two other sets of values, each set having a common factor. When $a=3$, $b=4$, $c=5$. When $a=4$, $b=3$, $c=5$. Multiplying these sets by the respective numbers that produce $a=12$, we obtain the required sets, 12, 16, 20; and 12, 9, 15, making in all 4 sets.

Put $a=15$. Then $mn=15=15\times1$ and 5×3. There are, therefore, two sets of prime integral values: 15, 112, 113; and 15, 8, 17. But as $15=5\times3$ and 3×5, there are also two sets of values, each set having a common factor. When $a=3$, $b=4$, $c=5$. When $a=5$, $b=12$, $c=13$. Whence the required sets are 15, 20, 25; and 15, 60, 65,—in all 4 sets.

In order to find the number of sets of values that can be formed for a=an integer, we shall illustrate by taking $a=60$. Then $2mn=60=2\times30\times1$, $2\times15\times2$, $2\times10\times3$, and $2\times6\times5$. Hence there are 4 sets of prime integral val-

ues. But 60 contains also the following factors that are odd numbers: $3=3\times1$; $5=5\times1$; and $15=15\times1$ and 5×3. These give 4 more sets. The factors that are even numbers divisible by 4, are $4=2\times2\times1$; $12=2\times6\times1$ and $2\times3\times2$; and $20=2\times10\times1$ and $2\times5\times2$. These give 5 additional sets. Hence for $a=60$, there are 13 sets of values for integral sides of right triangles.

A THEOREM ON PRISMOID.

By P. H. PHILBRICK, C. E., Pineville, Louisiana.

THEOREM. *To prove that the error of the "end area volume" of any prismoid or solid to which the prismoidal formula applies, is twice the error of the "middle area volume" and on the opposite side of the true result.*

Let A and B represent the end areas, M the middle area, and l the length of the prismoid.

Then the true volume is, $V=\frac{1}{6}l(A+4M+B)$.....(1),

the end area volume is, $V_e=\frac{1}{2}l(A+B)$.....(2),

and the middle area volume is, $V_m=lM$.....(3).

Now (1)$-$(2) gives error of (2)$=V-V_e=\frac{1}{6}l(4M-2A-2B)$.....(4),

and (1)$-$(3) gives error of (3)$=V-V_m=\frac{1}{6}l(A+B-2M)$.....(5).

But (4) is twice (5) with a contrary sign.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by JOHN T. FAIRCHILD, Principal of Crawfis College, Crawfis College, Ohio.

When U. S. bonds are quoted in London at 108¾ and in Philadelphia at 112½, exchange \$4.89½, gold quoted at 107, how much more was a \$1000 U. S. bond worth in London than in Philadelphia?

No solution of this problem has been received.

75. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

If 24 men, in 15 days of 12 hours each, dig a trench 300 yards long, 5 yards wide, 6 feet deep for 540 five-cent loaves when flour is $8 a barrel; what is flour worth a barrel when 45 men, working $5\frac{1}{3}$ days of ten hours each, dig a trench 125 yards long, 5 yards wide, 8 feet deep for 320 four-cent loaves? *Solve by proportion.*

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and C. A. JONES, Terrence, Miss.

The price of flour is an inverse ratio, hence using the *cause* and *effect* process we get at once

$$\left\{\begin{matrix}24\\15\\12\\540\\5\end{matrix}\right\} : \left\{\begin{matrix}300\\5\\6\\(?)\end{matrix}\right\} :: \left\{\begin{matrix}45\\5\frac{1}{3}\\10\\320\\4\end{matrix}\right\} : \left\{\begin{matrix}125\\5\\8\\8\end{matrix}\right\}$$

$$\therefore (?) = \frac{24\times15\times12\times540\times5\times125\times5\times8\times8\times3}{300\times5\times6\times45\times16\times10\times320\times4} = 16\tfrac{7}{8}.$$

$\therefore$ Flour is worth $16.87\frac{1}{2}$ per barrel.

76. Proposed by E. W. MORRELL, Professor of Mathematics in Montpelier Seminary, Montpelier, Vermont.

An eastern nobleman willed his entire estate to his three sons on the condition that the oldest should have one-half, the next one-third, and the youngest one-ninth. His estate, on inventory, was found to consist of 17 elephants. What should be the share of each?

Solution by FREDERIC R. HONEY, Ph. B., New Haven, Connecticut, and CHAS. C. CROSS, Laytonsville, Maryland.

If the will was obeyed literally the eldest son's share was $_{2}^{17}$ elephants; the second son's $_{3}^{17}$; and the youngest's $_{9}^{17}$, making a total of $_{2}^{17}+{}_{3}^{17}+{}_{9}^{17}=16\,{}_{18}^{1}$ elephants. This would leave $_{18}^{17}$ of an elephant.

The following solution would be satisfactory:

We have $\frac{1}{2}+\frac{1}{3}+\frac{1}{9}=\frac{17}{18}$ as the denominator.

First son receives $\frac{\frac{1}{2}}{\frac{17}{18}}=\frac{1}{2}\times\frac{18}{17}=\frac{9}{17}$.

Second son receives $\frac{\frac{1}{3}}{\frac{17}{18}}=\frac{1}{3}\times\frac{18}{17}=\frac{6}{17}$.

Third son receives $\frac{\frac{1}{9}}{\frac{17}{18}}=\frac{1}{9}\times\frac{18}{17}=\frac{2}{17}$.

Since the estate consisted of 17 elephants, $\therefore$ the first son got $_{17}^{9}$ of $17=9$ elephants; and the second son got $_{17}^{6}$ of $17=6$ elephants; and the third son got $_{17}^{2}$ of $17=2$ elephants.

Remarks by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

As 17 is prime, the elephants should be divided as near the proportion as possible. $\therefore$ oldest should have 9, next, 6, and the youngest, 2.

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

68. Proposed by LEONARD E. DICKSON, M. A., Ph. D., Formerly Fellow of Mathematics, University of Chicago, Chicago, Illinois.

Suppose a circle of unit radius divided at the points A, A_1, A_2, A_3, ... into n equal parts. [This division cannot in general be affected by geometry.] Through A draw the diameter OA and join O with $A_1, A_2, A_3, \ldots A_{\frac{n-1}{2}}$, where n is supposed to be odd.

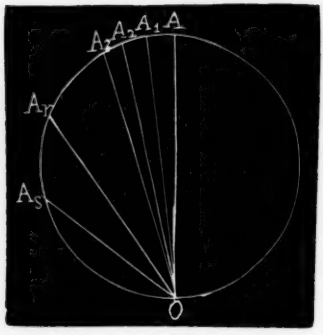
Prove that $OA_1 - OA_2 + OA_3 - OA_4 + \ldots\ldots \pm OA_{\frac{n-1}{2}}$, every other chord being affected with the minus sign.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas, and C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Let OA_1, OA_2, OA_3, etc. $= a_1, a_2, a_3$, etc.

Now $OA = 2$, $\angle AOA_1 = \angle A_1OA_2 = \text{etc.} = \pi/n$.

$\therefore\ OA_r = a_r = 2\cos(r\pi/n)$.



(1) When $\frac{n-1}{2}$ is even,

$$\therefore\ a_1 + a_3 + a_5 + \ldots\ldots + a_{\frac{n-3}{2}}$$

$$= 2\left(\cos\frac{\pi}{n} + \cos\frac{3\pi}{n} + \cos\frac{5\pi}{n} + \ldots\ldots\right.$$

$$\left. + \cos\frac{(n-3)\pi}{2n}\right) = \sin\frac{(n-1)\pi}{2n} / \sin\frac{\pi}{n}$$

$$= \sin\left(\frac{\pi}{2} - \frac{\pi}{2n}\right) / \sin\frac{\pi}{n} = \tfrac{1}{2}\cos\frac{\pi}{2n} \quad \ldots\ldots\ (1).$$

$$a_2 + a_4 + a_6 + \ldots\ldots + a_{\frac{n-1}{2}} = 2\left(\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \ldots. + \cos\frac{(n-1)\pi}{2\pi}\right)$$

$$= \frac{2\cos[(n+3)\pi/4n]\sin[(n-1)\pi/4n]}{\sin(\pi/n)} = \frac{2\cos[\frac{1}{4}\pi + (3\pi/4n)]\sin[\frac{1}{4}\pi - (\pi/4n)]}{\sin(\pi/n)}$$

$$= \frac{[\cos(3\pi/4n) - \sin(3\pi/4n)][\cos(\pi/4n) - \sin(\pi/4n)]}{\sin(\pi/n)} = \frac{\cos(\pi/2n) - \sin(\pi/n)}{\sin(\pi/n)}$$

$$= \tfrac{1}{2}\operatorname{cosec}\frac{\pi}{2n} - 1 \ldots\ldots \quad \ldots\ldots (2).$$

$$\therefore a_1-a_2+a_3-a_4+\ldots\ldots+a_{\frac{n-3}{2}}-a_{\frac{n-1}{2}}=1.$$

(2) When $\frac{n-1}{2}$ is odd.

$$\therefore a_1+a_3+a_5+\ldots\ldots+a_{\frac{n-1}{2}}=2(\cos\frac{\pi}{n}+\cos\frac{3\pi}{n}+\cos\frac{5\pi}{n}+\ldots\ldots$$

$$+\cos\frac{(n-1)\pi}{2n})=\sin\frac{(n+1)\pi}{2n}/\sin\frac{\pi}{n}$$

$$=\sin\left(\frac{\pi}{2}+\frac{\pi}{2n}\right)/\sin\frac{\pi}{n}=\tfrac{1}{2}\operatorname{cosec}\frac{\pi}{2n}\ldots\ldots\ldots\ldots(3).$$

$$a_2+a_4+a_6+\ldots\ldots+a_{\frac{n-3}{2}}=2(\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cos\frac{6\pi}{n}+\ldots\ldots+\cos\frac{(n-3)\pi}{2n})$$

$$=\frac{2\cos[(n+1)\pi/4n]\sin[(n-3)\pi/4n]}{\sin(\pi/n)}=\frac{2\cos[\frac{1}{4}\pi+(\pi/4n)]\sin[\frac{1}{4}\pi-(3\pi/4n)]}{\sin(\pi/n)}$$

$$+\frac{[\cos(n/4n)-\sin(\pi/4n)][\cos(3\pi/4n)-\sin(3\pi/4n)]}{\sin(\pi/n)}=\frac{\cos(\pi/2n)-\sin(\pi/n)}{\sin(\pi/n)}$$

$$=\tfrac{1}{2}\operatorname{cosec}(\pi/2n)-1.\ \ldots\ldots\ldots\ldots(4).$$

$$\therefore a_1-a_2+a_3-a_4+\ldots\ldots-a_{\frac{n-3}{2}}+a_{\frac{n-1}{2}}=1.$$

$$\therefore OA_1-OA_2+OA_3-OA_4+\ \ldots\ldots\pm OA_{\frac{n-1}{2}}=1.$$

70. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

Prove that the locus of the center of the circle which passes through the vertex of a parabola and through its intersections with a normal chord is the parabola $2y^2=ax-a^2$, the equation to the given parabola being $y^2=4ax$.

Solution by the PROPOSER.

The circles being $(x-m)^2+(y-n)^2=r^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$,

and passing through the vertex of $y^2=4ax\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$,

becomes $x^2-2mx+y^2-2ny=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$.

Now the extremities of the normal chord being $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, normal at the former point, we have

$$a^2t_1^4-2mt_1^2+4a^2t_1^2-4nat_1=0\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\text{and } a^2t_2^4-2mt_2^2+4a^2t_2^2-4nat_2=0\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Divide these by at_1, at_2 respectively, and take one result from the other and divide by t_1-t_2; then

$$a(t_1^3+t_1^2t_2+t_1t_2^2)=2mt_1-4at_1 \quad \ldots\ldots(6).$$

Divide (4) by at_1 and take (6) from the result; then

$$at_1t_2(t_1+t_2)=-4n \quad \ldots\ldots(7).$$

But it can be shown that $t_1+t_2=-(2/t_1)$ $\ldots\ldots(8)$.

Substituting in (7) and reducing, $t_2=(2n/a)=-t_1-(2/t_1)$ $\ldots\ldots(9)$,

which gives $t_1=\dfrac{-n+\sqrt{n^2-2a^2}}{a}$ $\ldots\ldots(10)$.

(10) in (6) gives $2n^2=am-a^2$ $\ldots\ldots(11)$,

the required locus of center of (1).

Also solved by *F. M. McGAW*.

[NOTE. Solution of Problen 69 will appear in the next issue. Editor.]

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

There is a triangle whose sides repulse a center of force within the triangle with an intensity that varies inversely as the distance of the center of force from each point of the sides of the triangle. What the position of equilibrium of the center ?

Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

Put p=altitude of the triangle upon the side a as base, s=distance of center of force from the side a, x=distance of any point of side a from the vertex B, y=distance of any point of the side b from the vertex C, and z=distance of any point of the side c from the vertex B.

The force exerted by any portion dx of the side a resolved perpendicular to it, is $msdx$ where m is an arbitrary constant depending on the intensity of the force. The forces exerted by portions dy and dz of the sides b and c are respectively $m[s-(b/p)y]dy$ and $m[s-(c/p)z]dz$. For equilibrium we have

$$m\int_0^a s\,dx+m\int_0^b(s-\frac{p}{b}y)dy+m\int_0^c(s-\frac{p}{c}z)dz=0.$$

$$\therefore (a+b+c)s=\tfrac{1}{2}p(b+c). \quad \therefore s=\frac{p(b+c)}{2(a+b+c)}.$$

Hence the distance of the center of force from the side a is $\frac{p(b+c)}{2(a+b+c)}$.

In like manner it may be shown that its distance from the side b is $\frac{q(a+c)}{2(a+b+c)}$, and that its distance from the side c is $\frac{r(a+b)}{2(a+b+c)}$, when q and r are the altitudes of the triangle upon the sides b and c respectively.

Also solved by *G. B. M. ZERR*. His solution will appear in the next issue.

45. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A fifty-pound cannon-ball is projected vertically upward with a velocity of 300 feet per second. Find the height to which it will rise and the time of flight, assuming the initial resistance of the air on the ball to be 10 pounds and the resistance to vary as the square of the velocity.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let h=height required, t=time of ascent, t_1=time of descent, $T=t+t_1$= time of flight, v=velocity=300 feet per second. W=50 pounds, g=32.2 feet per second. μv^2=10 pounds=$\frac{1}{5}W$.

$$\therefore \mu=\frac{W}{5v^2}. \quad \frac{1}{k}=\sqrt{\frac{W}{\mu}}=v\sqrt{5}. \quad \therefore k=\frac{1}{v\sqrt{5}}.$$

From Bowser's Analytical Mechanics, pages 306-7, we get,

$$h=\frac{1}{2gk^2}\log(1+k^2v^2), \quad t=\frac{1}{gk}\tan^{-1}vk.$$

$$t_1=\frac{1}{gk}\log\{\sqrt{1+v^2k^2}+vk\}.$$

$$\therefore h=\frac{5v^2}{2g}\log(\tfrac{6}{5})=1273.9827 \text{ feet}.$$

$$t=\frac{v\sqrt{5}}{g}\tan^{-1}\frac{1}{\sqrt{5}}=8.75931 \text{ seconds}.$$

$$t_1=\frac{v\sqrt{5}}{g}\log\left(\frac{\sqrt{6}+1}{\sqrt{5}}\right)=9.03118 \text{ seconds}.$$

$$T=t+t_1=17.79049 \text{ seconds}.$$

Also solved by *HENRY HEATON*.

46. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"There was an old woman tossed up in a basket
Ninety times as high as the moon." *Mother Goose.*

Neglecting the resistance of the air, how long did it take the old lady to go up?

I. Solution by E. L. SHERWOOD, A. M., Superintendent of City Schools, West Point, Mississippi.

The equation of motion is

1. $\frac{d^2s}{dt^2} = -\frac{gr^2}{s^2}$. 2. $\left(\frac{ds}{dt}\right)^2 = \frac{2gr^2}{s} + c$

where ds/dt or $v=0$, when $S=90\times 60.3R$ or $5427R$.

Whence $C=-\frac{2gr^2}{5427R}$.

3. $\left(\frac{ds}{dt}\right)^2 = \frac{2gr^2}{s} - \frac{2gr^2}{5427r}$ or $2gr^2\left(\frac{1}{s}-\frac{1}{a}\right)$.

4. $dt=\sqrt{\frac{a}{2gr^2}}\cdot\frac{sds}{\sqrt{as-s^2}}$ solving dt in (3).

5. $t=\sqrt{\frac{a}{2gr^2}}\int_a^R \frac{sds}{\sqrt{as-s^2}}$ for $t=0$ when $s=a$.

6. $t=\sqrt{\frac{a}{2gr^2}}\left[\left(\sqrt{as-s^2} - \tfrac{1}{2}a\operatorname{vers}^{-1}\frac{2s}{a}+C\right)\right]_a^R$.

7. $t=\sqrt{\frac{a}{2gr^2}}\left(\sqrt{aR-R^2} - \tfrac{1}{2}a\operatorname{vers}^{-1}\frac{2R}{a}+\frac{\pi a}{2}\right)$ where $a=5427R$.

8. $t=11.35+$ years, by substituting values and reducing.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let t=time, $R=3963$ miles$=20924640$ feet=radius of the earth, $g=32.2$ feet=gravity, $a=90(60R)=5400R$=distance the old woman was tosssd.

$$\therefore\ t=\sqrt{\frac{a}{2gR^2}}\left(\sqrt{aR-R^2}-\tfrac{1}{2}a\operatorname{vers}^{-1}\frac{2R}{a}+\frac{\pi a}{2}\right).$$

$$t=30\sqrt{\frac{3}{gR}}\{R\sqrt{5399}-2700R\operatorname{vers}^{-1}{}_{\frac{1}{2700}}+2700\pi R\}.$$

t=355287708.316 seconds=11 years, 3 months, 7 days, 3 hours, 1 minute, 48.316 seconds.

Also solved by *J. C. CORBIN.*

47. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in Columbian University, Washington, D. C.

What is the focus of the convex surface of a plano-convex lens, index μ, which will converge parallel monochromatic rays to a given focus, the rays entering the lens on the plane side?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let f=the given focal length. F=the focal length required,

u=distance of origin of ray from lense,

r, s, the radii of the first and second surfaces of the lense respectively,

t=the thickness, and regard all distances as measured from the posterior surface.

Then we have for for a double convex lense,

$$\frac{1}{\frac{1}{f}+\frac{\mu-1}{s}}-\frac{1}{\frac{1}{u}-\frac{\mu-1}{r}}=\frac{t}{\mu}.$$

(See Parkinson's Optics, Art. 100, Cor. I, page 91).

Let $u=r=\infty$.

$$\therefore \frac{1}{\frac{1}{f}+\frac{\mu-1}{s}}=\frac{t}{\mu} \quad \ldots\ldots\ldots\

48. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Two equal heavy rings connected by a string passing over a peg at the focus of a conic section will be in equilibrium at all points on the curve.

Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics, Columbian University, Washington, D. C.

An evident property of *any* curve which will be a curve of equilibrium for two weights thus attached is that the tension along the string shall be the same, wherever the weights are placed upon the curve. If this were not so, by altering the position of one of them, by changing the length of the string, the tension would be changed and the other would no longer be in equilibrium.

Call T the tension, W the weight, ϕ the angle which the curve makes with the horizontal, and θ the angle which the string makes.

Resolving along the curve,

$$W\cos\phi - T\cos(\theta-\phi) = 0.$$

$$\frac{\cos(\theta-\phi)}{\cos\phi} = \frac{W}{T} = k.$$

$\therefore\ \cos\theta + \sin\theta\tan\phi = k.$

$$\frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}} \cdot \frac{dy}{dx} = k. \qquad \frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}} = k\,dx.$$

$$(x^2+y^2)^{\frac{1}{2}} = kx+c. \qquad x^2(1-k^2)+y^2-2kcx=c^2.$$

This is the equation of a conic with the origin at the focus.

$$k = e+c = a(1-e^2).$$

The above investigation refers to the case in which the tension is simply constant. The string may be attached to the fixed point.

If the string be now considered passing around the focus to the curve again and a weight W attached there also, the tension will be doubled.

Then $k = W/2T$.

If $W=2T$, or $T=\frac{1}{2}W$, the equation becomes $y^2-2cx=c^2$, a parabola.

If $W>2T$, or $T<\frac{1}{2}W$, the curve is an hyperbola.

If $W<2T$, or $T>\frac{1}{2}W$, the curve is an ellipse.

The above may be put in the following form:

In a parabola the tension is equal to half one of the equal weights; in an hyperbola it is less than half of the weight; and in an ellipse it is greater than the same.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

43. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

In a circle whose radius is a, chords are drawn through a point distant b from the center. What is the average length of such chords, (1), if a chord is drawn from every point of the circumference, and (2), if they are drawn through the point at equal angular intervals?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

Let $AC=CF=a$, $CD=b$, $\angle FDA=\theta$, $\angle FCA=\varphi$, $\tan\theta=m$.

Then the equation to DF is, $y=mx+mb$ .....(1).

The equation to the circle is, $x^2+y^2=a^2$ ....(2).

From (1) and (2) we easily get $EF=2\sqrt{a^2-\dfrac{m^2b^2}{1+m^2}}$,

$\therefore EF=2\sqrt{a^2-b^2\sin^2\theta}$. But $\sin\theta=\dfrac{a\sin\varphi}{\sqrt{a^2+b^2+2ab\cos\varphi}}$.

$$\therefore EF=\frac{2(a^2+ab\cos\varphi)}{\sqrt{a^2+b^2+2ab\cos\varphi}}=\frac{2(a^2+ab-2ab\sin^2\frac{1}{2}\varphi)}{\sqrt{(a+b)^2-4ab\sin^2\frac{1}{2}\varphi)}}.$$

The limits of φ for $b>a$, are 0 and $\frac{1}{2}\pi+\sin^{-1}(a/b)=2\beta$.

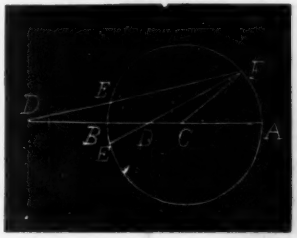
The limits of φ for $b<a$, are 0 and $\frac{1}{2}\pi+\sin^{-1}(b/a)=2\delta$.

The limits of θ for $b>a$, are 0 and $\sin^{-1}(a/b)=\theta'$.

The limits of θ for $b<a$, are 0 and $\frac{1}{2}\pi$.

Let Δ and Δ_1 be the average lengths required.

$$\therefore \Delta=\int EFds/\int ds=\int EFd\varphi/\int d\varphi,$$



$$\Delta_1=\int EFd\theta/\int d\theta.$$

I. Let $\dfrac{4ab}{(a+b)^2}=e^2$, and $\frac{1}{2}\varphi=\gamma$.

$\therefore d\varphi=2d\gamma$.

Then $EF=\dfrac{2(a^2+ab-2ab\sin^2\gamma)}{(a+b)\sqrt{1-e^2\sin^2\gamma}}$.

$$\therefore\ EF=\frac{2a}{\sqrt{1-e^2\sin^2\gamma}}+\frac{2}{e\sqrt{ab}}\sqrt{1-e^2\sin^2\gamma}-\frac{2}{e\sqrt{ab}\sqrt{1-e^2\sin^2\gamma}}.$$

$$\therefore\ \int EF\,d\gamma=\frac{2}{e\sqrt{ab}}\{(ae\sqrt{ab}-1)F(e,\gamma)+E(e,\gamma)\}.$$

$$\therefore\ J=\frac{2}{\beta e\sqrt{ab}}\{(ae\sqrt{ab}-1)F_0^{\beta}(e,\gamma)+E_0^{\beta}(e,\gamma)\},\ b>a.$$

$$J=\frac{2}{\delta e\sqrt{ab}}\{(ae\sqrt{ab}-1)F_0^{\delta}(e,\gamma)+E_0^{\delta}(e,\gamma)\},\ b<a.$$

II. $J_1=2\int_0^{\theta'}\sqrt{a^2-b^2\sin^2\theta}\,d\theta\Big/\int_0^{\theta'}d\theta=\dfrac{2}{\theta'}\int_0^{\theta'}\sqrt{a^2-b^2\sin^2\theta}\,d\theta.$

Let $\theta=\frac{1}{2}\pi+\lambda$. $\therefore\ \theta'-\frac{1}{2}\pi=\lambda$ to $-\frac{1}{2}\pi=\lambda$.

$$\therefore\ J_1=\frac{2}{\theta'}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\sqrt{a^2-b^2\cos^2\lambda}\,d\lambda=\frac{2}{\theta'}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\sqrt{b^2\sin^2\lambda-(b^2-a^2)}\,d\lambda$$

$$=\frac{2\sqrt{(b^2-a^2)}}{\theta'}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\sqrt{\frac{b^2}{b^2-a^2}\sin^2\lambda-1}\,d\lambda$$

$$=\frac{2\sqrt{(b^2-a^2)}}{\theta'}H_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi}\left(\frac{b}{\sqrt{b^2-a^2}},\lambda\right),\ b>a.$$

$$J_1=2\int_0^{\frac{1}{2}\pi}\sqrt{a^2-b^2\sin^2\theta}\,d\theta\Big/\int_0^{\frac{1}{2}\pi}db=\frac{4a}{\pi}E_0^{\frac{1}{2}\pi}\left(\frac{b}{a},\ \ \right),\ b<a.$$

45. Proposed by J. C. WILLIAMS, Boston, Massachusetts.

At the end of the fifth inning the base ball score stands 7 to 9. What is the probability of winning for either team?

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

From the stated score we are able to estimate the respective skill of the two teams, and their respective probabilities of winning the game.

The respective probabilities are $\frac{7}{16}$ and $\frac{9}{16}$. We have now to find the probabilities of either team winning *at least* 3 games out of 4, granting, of course, 9

innings to be played. These probabilities are respectfully, $(\frac{7}{16})^4+4(\frac{7}{16})^3\cdot\frac{9}{16}=\frac{14}{65}\frac{7}{53}\frac{4}{8}\frac{9}{8}$, and $(\frac{9}{16})^4+4.(\frac{9}{16})^3\cdot\frac{7}{16}=\frac{26}{65}\frac{9}{53}\frac{7}{8}\frac{3}{8}$.

46. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Four men starting from random points on the circumference of a circular field and traveling at different rates, take random straight courses across it; find the chance that at least two of them will meet.

Professor Heaton says: "If the men are considered points the chance is 0." [A possible though difficult problem could be made of this one by using instead of men segments of straight lines moving along random secants of a circle, the velocity of the segments all being different. Editor.]

47. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of the chords that may be drawn from one extremity of the major axis of an ellipse to every point of the curve?

Solution by the PROPOSER.

The length of a single chord is

$$[(a-x)^2+y^2]^{\frac{1}{2}}=(1/a)[a^2(a^2-x^2)+b^2(a^2-x^2)]^{\frac{1}{2}}.$$

Put S=distance around the ellipse. Then the required average is $A=$

$$\frac{2}{aS}\int_0^{\frac{1}{4}S}[a^2(a-x)^2-b^2(a^2-x^2)]^{\frac{1}{2}}\,dS=$$

$$\frac{2}{a^2S}\int_{-a}^{+a}\frac{[a^2(a-x)^2+b^2(a^2-x^2)]^{\frac{1}{2}}\,[a^2(a^2-x^2)+b^2x^2]^{\frac{1}{2}}\,dx}{(a^2-x^2)^{\frac{1}{2}}}=$$

$$\frac{2}{a^2S}\int_{-a}^{+a}\frac{[a(a^2+b^2)-(a^2-b^2)x]^{\frac{1}{2}}\,[a^4-(a^2-b^2)x^2]^{\frac{1}{2}}\,dx}{(a+x)^{\frac{1}{2}}}.$$

This is readily reducible to elliptic functions of the first and second order, but the expressions I have been able to obtain are involved radicals.

Also solved by *G. B. M. ZERR* and *J. F. SCHEFFER*.

NOTE ON PROBLEM 391

BY LEWIS NEIKIRK, BOULDER, COLORADO.

The man starts at O moving in a *perfectly random* manner. After t seconds suppose him at P and that during the next instant dt he travels through ds to m at an angle θ with the line OP. Let $PM=dr=ds\cos\theta=v\cos\theta dt$, since $ds=vdt$. He will escape from the desert if $\int dr>R$ (the radius) the limits of inte-

gration being those which correspond to 0 and T of t; that is, if $\int_0^T v\cos\theta dt > R$. But this integral depends upon two independent variables. Indeed, θ, being wholly discontinuous from point to point according to the conditions of the problem, can not be considered a variable at all. If however, we assume θ constant (i. e. if the "perfectly random" motion of the problem means motion in a logarithmic spiral) then the condition above reduces to $vT\cos\theta > R$; or $\theta > \cos^{-1}(R/vT)$, agreeing with Professor Anthony.

NOTE ON PROBLEM 39.

BY J. BURKETT WEBB, C. E., PROFESSOR OF MATHEMATICS AND MECHANICS, STEVENS INSTITUTE OF TECHNOLOGY, HOBOKEN, NEW JERSEY.

It seems to me that every such problem should have a complete and intelligible *physical* idea behind it, and further that a solution should be a development of the *physical ideas* of the problem, mathematics being simply the grammatical language of physics.

If Professor Anthony has a complete idea in the problem it is not intelligible to me and so it may be best to state the difficulties which appear to me.

It is to be inferred from the solution that the "perfectly random manner" means that the path consists of differential elements of equal length and all possible directions arranged in a chance succession.

If so the man will never reach the edge of the desert, or, stated otherwise, he will have but one chance in an infinite number of doing so.

In the solution the *rate of approach to the circumference* is spoken of; in random movements there would be no such rate except as the average of actual rates and this is not the use made of it.

The solution also supposes the man at each instant to go *within the angle MPK*, but this he does not need to do to get off in the time; so the deduced chance seems not to follow.

In fact the chance C=etc., is the answer to a different problem, as I see the matter, namely: Of all logarithmic spirals joining the center and circumference, having their origins at the center of the circle and differing from each other by equal increments of the angle between the radius vector and curve, what is the chance of choosing at random one whose included arc shall be less than Tv?

To make the problem apply to the case, for which it was I suppose, intended, of a wanderer in a desert I think one of two things will be needed. Either a certain finite length of step, taken at random must be fixed, or a law established to make large changes of direction less likely than small ones.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

77. Proposed by F. S. ELDER, Professor of Mathematics, Oklahoma University, Norman, Oklahoma.

For how many seconds must I count the clicking of the rails under a train that the number of rails counted may be equal to the speed of the train in miles per hour, a rail being 30 feet long?

78. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, New Jersey.

Solve by pure arithmetic, no algebraic symbols: A Texan farmer owns 5169 cattle; there are 3 times as many horses as cows, plus 569, and 4 times as many cows as sheep, minus 126; how many has he of each? [From *Brooks' Higher Arithmetic.*]

79. Proposed by F. M. PRIEST, St. Louis, Missouri.

How many $20 gold pieces can be put in a room 20 feet long, 18 feet wide, and 9 feet high?

GEOMETRY.

77. Proposed by CHARLES C. CROSS, Laytonsville, Maryland.

A line is drawn perpendicular to BC, of the triangle ABC, whose sides are $BC=a$, $CA=b$, and $AB=c$, through A to D, a distance d, (d being equal to or greater than $a+b$); from D a line is drawn to E, a distance e, (e being equal to or greater than $a+b+c$) on BC extended. Required the area of the ellipse which is isogonal conjugate to the straight line DE with respect to the triangle ABC.

78. Proposed by J. A. MOORE, Professor of Mathematics, Millsaps College, Jackson, Mississippi.

Required the number of normals that can be drawn from any point (a, b) to the parabola $y^2 = 2px$.

77. Proposed by JOHN MACNIE, Professor of Mathematics, University of North Dakota, University, North Dakota.

To construct a quadrilateral of given area, the diagonals, one of which is given, cutting each other in given ratios and at a given angle.

MECHANICS.

55. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Mississippi.

Three equal heavy spheres, each of weight W, are placed on a rough ground just not touching each other. A fourth sphere of weight nW is placed on the top touching all three. Show that there is equilibrium if the coefficient of friction between two spheres is greater than $\tan\frac{1}{2}\alpha$, and that between a sphere and the ground is greater than $\tan\frac{1}{2}\alpha n/(n+3)$, where α is the inclination to the vertical of the straight line joining the centers of the upper and one lower sphere.

56. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"Hey-diddle-diddle, the cat and the fiddle,
The cow jumped over the moon."

Taking the weight of the cow to be 600 pounds, the initial resistance of the air to be 100 pounds and varying as the square of the velocity, find the initial and final velocities and the times of rising and falling.

57. Proposed b J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas, College Station, Texas.

Over the intersection of two inclined planes slides a cord of uniform mass throughout its length. Find the equation to the path described by its center of gravity.

AVERAGE AND PROBABILITY.

54. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

A man is at the center of a circle whose diameter is equal to three of his steps. If each step is taken in a perfectly random direction, what is the probability, (1), that he will step outside the circle at the second step, and, (2), that he will step outside at the third step?

55. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas.

It has been clear for 15 consecutive days, what is the chance of the 16th day being cloudy?

56. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the chance that the center of gravity of a triangle lies inside the triangle formed by three points taken at random within the triangle. [From *Williamson's Integral Calculus.*]

NOTES.

NOTE ON MR. BECHER'S ARTICLE IN OCTOBER NUMBER OF MONTHLY.

BY J. R. BALDWIN, DAVENPORT, IOWA.

In Franklin A. Becher's article for the October number, (Vol. III), I notice he says, "Multiplying an infinite number by another gives us infinity of a higher power, or dividing gives us infinity of a lower power."

How does he reconcile the latter part of this statement with Wallis's expression for the value of π,

$$\tfrac{1}{2}\pi = \frac{2.2.4.4.6.6.8.8 \ldots\ldots}{1.3.3.5.5.7.7.9 \ldots\ldots}\ ?$$

In this expression, we have the quotient of two infinite numbers equal to a finite number.

Mr. Lilley's criticism (MONTHLY, Vol. III., No. 3.) of the solution IV. (MONTHLY, Vol. II., page 190) is undoubtedly valid, but the statement that "Todhunter failed to produce a direct proof of it" is probably incorrect. The theorem is given by Todhunter (Euclid, page 316) and in a note at the bottom of page 317 he says, "For the history of this theorem see *Lady's and Gentlemen's Diary* for 1859, page 88." If any reader of the MONTHLY has the Diary for that year I should be very much pleased to see the history of this theorem published in the MONTHLY. Todhunter's proof of the theorem is indirect, but that does not argue that he was unable to discover a direct proof. I remember that many

years ago when reading Todhunter's Euclid I attempted a direct proof of this theorem but failed. The proof on page 157, Vol. II. of the MONTHLY is a direct proof and, with the exception of a few mistakes in lettering, seems to be free from objection. A slight simplification may be made by proving the equality of the triangles *ADB*, *BFA*, instead *ADF*, *BDF*. WM. E. HEAL.

MULTI-DIRECTIONAL GEOMETRY.

BY JOHN N. LYLE, PH. D., BENTONVILLE, ARKANSAS.

The concept plane, rectilineal angle implies that there are straight lines and also, that they are located in different directions.

Hence, no system of plane geometry or of spherical geometry for that matter, is free from assumptions regarding "direction."

In some geometrical systems, however, larger use is made of "direction," both word and thing, than in others. Euclid, by his three geometrical axioms and his three postulates places restriction upon "directional geometry" which cannot be relaxed without endangering these axioms and postulates.

According to the Euclidean geometry there is but *one straight path* from the point *A* to the point *B*. That path marks the direction from *A* to *B*. A body moving in this direction on this path approaches *B* until *B* is reached. A body moving in the opposite direction along the same straight path recedes farther and farther from *B*.

This is the pure Euclidean doctrine, clear and strong, free from the suspicion even of a hypothetical "point at infinity" where ungeometrical deeds are reported to be done.

According to the Euclidean view, then, there is but one direction from *A* to *B*. According to Olans Henrici's view as given in the Article on Geometry in the Ninth Edition of the Encyclopœdia Britannica there are at least two directions diametrically opposite to each other from *A* to *B*; one direct and finite in length; the other roundabout via "the point at infinity."

This latter route can hardly be called "air line." Let us notice just one logical difficulty. Every path that reaches *B* drawn from *A* must be continuous. But a continuous line with two ends *A* and *B* must be finite. Hence, the hypothesis that a continuous line, infinite in length, can be drawn between two points *A* and *B* is a flagrant violation of the logical law of Non-Contradiction. By the way, this logical law is the bed rock on which the *reductio ad absurdum* process of reasoning is founded.

Another species under the genus Directional hypothesis is the Multi-Directional hypothesis. According to this hypothesis *B* may be so located with respect to *A* that myriads of different straight lines may be drawn between the two points. That is, *B* is myriads of different directions from *A*. This result seems to me to be absurd. Hence, for that reason, I would reject it. Many modern mathematicians, however, regard their hypotheses as beyond the reach of *reductio ad absurdum* method and the fundamental laws of thought.

EDITORIALS.

Professor Colaw was called away from home during the greater part of the past month, which fact will explain why his departments have been omitted in this issue.

We are happy to announce that a series of short elementary expository articles on Lie's Transformation Groups by Dr. E. O. Lovett, Baltimore, Maryland, will begin in the May number.

Professor Ollis Howard Kendall died last week at his home in Philadelphia. Professor Kendall was for a number of years Assistant Professor of Mathematics at the University of Pennsylvania, at the same time that his father occupied the Chair of Mathematics.

BOOKS AND PERIODICALS.

Algebra Reviews. By Edward Rutledge Robbins, Master in Mathematics and Physics, The Lawrenceville School. Paper Back, 44 pages. Chicago: Ginn & Co.

The object of this little book is to present the essentials of Elementary Algebra in a form sufficiently complete as to be helpful to teachers and students at the time of review. The exercises are various and well selected. Teachers desiring such a book, will find this one well suited to their needs. B. F. F.

Thoughts on Religion. By the late George John Romanes, M. A., LL. D., F. R. S., Canon of Westminster. Edited by Charles Gore, M. A., Canon of Westminster. Cloth, gilt top, 184 pages. Price, $1.25. Chicago: The Open Court Publishing Co.

The value and importance of this work on the thought and conscience of the world cannot be overestimated. Coming as it does from one of the foremost agnostics and scientific thinkers of his time, it comes as a revelation to all classes of readers. In this book can be studied the evolution of a master mind from adhering to the doctrine of agnosticism to that of a full acceptance of the religion of Jesus Christ. B. F. F.

Darwin, and After Darwin. An Exposition of the Darwinian Theory and a Discussion of the Post-Darwinian Questions. By George John Romanes, M. A., LL. D., F. R. S., Honorary Fellow of Gonville and Caius College, Cambridge. I. The Darwinian Theory. Second Edition. Cloth, gilt top, xiv and 460 pages. Price, $2.00. Chicago: The Open Court Publishing Co.

The first volume contains ten chapters. Chapter I, Introductory; Chapter II, Classification; Chapter III, Morphology; Chapter IV, Embryology; Chapter V, Palæontology; Chapter VI, Geographical Distribution; Chapter VII, The Theory of Natural Selection;

Chapter VIII, Evidences of the Theory of Natural Selection; Chapter IX, Criticisms of the Theory of Natural Selection; Chapter X, The Theory of Sexual Selection, and concluding remarks. A more earnest and convincing argument in favor of the Theory of Evolution has not appeared since Darwin's time. Dr. Romanes' grasp of thought and power of cogent reasoning appears in this volume with telling effect. No one with a fair knowledge of the methods of scientific investigations can fail, after having read this book, to be convinced of the truth of the theory. B. F. F.

University Algebra. By C. A. Van Velzer and Chas. S. Slichter, Professors in the University of Wisconsin. Pages 732. Madison, Wisconsin: Tracy, Gibbs and Company. 1893.

This book is now too well known to need any commendation from us. The authors are able and progressive teachers and in this text on algebra have introduced several new and valuable features. There are valuable chapters on mathematical induction, theory of limits, derivatives, complex numbers, the rational integral function, special equations, separation of roots, numerical equations, decomposition of rational fractions, graphic representation of equations, and determinants. The convergence and divergence of series is admirably treated. The accurate "historical notes" which are appended to the treatment of many of the topics will be appreciated. Every teacher of algebra has need of this work in his library whether he uses it as a class text-book or not. J. M. C.

Text-Book of Dynamics. University Tutorial Series. By William Briggs, M. A., F. C. S., F. R. A. S., and G. H. Bryan, M. A. Cloth, 105 pages. Price, 50 cents. Cambridge, England: W. B. Clive. New York Depot: Hinds & Noble, 4 Cooper Institute.

We called attention in a previous number to the text-book on *Hydro-Statics* by the same authors. The treatise on *Dynamics* deserves the same commendation. Due prominence is given to the principles of the subject, and in the solution of problems results are deduced as far as possible from these principles themselves. Worked examples are freely inserted, and hints relating to special difficulties are given where needed. The examples are numerous and practical, the examination papers well selected, and the summary of results after each chapter of special value in reviews. The book may be open to criticism on some minor points, but there are few text-books on this subject which are so well suited to the needs of beginners. J. M. C.

Theoretical Mechanics: Fluids. By J. Edward Taylor, M. A., B. Sc. 222 pages. Price, 80 cents. London and New York: Longmans, Green & Co.

Although intended to meet the Science and Art Department and London Matriculation requirements, this book may be used successfully in any school where a good text-book of its grade is required. One of the special features of the book is the large number of model examples which are fully worked out. The author believes they serve to fix the subject matter on the mind much more than simply reading over the text. This feature also makes it a valuable book to private students. The text is supplied with numerous graduated examples. J. M. C.

A Treatics on Elementary Hydrostatics. By John Greaves, M. A. Price, $1.10. 204 pages. Cambridge Press. New York: Macmillan & Co.

The author aims to treat the subject as fully as possible without using the Calculus, except in alternative proofs when by its aid results are more easily obtained or more concisely expressed. The mathematical element of the book is strong, and the book more advanced than the title and proposed method of treatment would indicate. It is well printed and furnished with sets of carefully selected exercises, while there are many excellent illustrative solutions. The topical index is helpful for ready reference. J. M. C.

Geometry of the Similar Figures and the Plane. By C. W. C. Barlow, M. A., B. Sc., and G. H. Bryan, M. A. Price, 60 cents. 123 pages. University Tutorial Series. Cambridge: W. B. Clive. New York Depot: Hinds & Noble.

This little book contains the Sixth and Eleventh Books of Euclid, together with a summary of Book V., and many important additional propositions and applications relating to the Geometry of Similar Figures and the Plane. Euclid's order has been closely followed, while the additional matter is mostly in the form of illustrative examples. The properties of centers of similitude and homologous points are collected in a supplement at the end of Book VI. In addition to the illustrative examples, numerous exercises for solution follow the propositions on which they depend. The feature of giving many alternative proofs enables the teacher to make his own choice of methods. It is a very satisfactory book in a useful series. J. M. C.

Modern Plane Geometry. By G. Richardson, M. A., and A. S. Ramsay, M. A. Price, $1.00. 202 pages. London and New York: Macmillan & Co.

This treatise includes chapters on properties of a triangle, quadrangle, and circle, harmonic and anharmonic ratio, geometrical maxima and minima, involution, reciprocation, inversion, and projection. . It gives all that is best in the recent geometry on these subjects and is an excellent introduction to the more advanced books of Cremona and others. In arrangement the sequence of propositions recommended by the Association for the improvement of Geometrical Teaching has been followed. The triangle has been very fully and satisfactorily treated. The book will serve as an excellent sequel to Euclid, and as a means of proceedure from Euclidean Geometry to the higher descriptive Geometry of Conics and of imaginary points. J. M. C.

Our Notions of Number and Space. By Herbert Nichols, Ph. D., assisted by W. E. Parsons, A. B. 201 pages. Price, $1.00. Ginn & Company, Boston.

This book is an experimental contribution to the "Genetic Theory of Mind." It aims to trace out the origin and development of our present perceptions of number and space from the nature of our past experiences. The experiments were conducted with great care and patience, and the results are worthy of being placed in this permanent and accessable form. The general survey and summary at the end of the book are helpful and valuable. J. M. C.

Business Forms, Customs, and Accounts. By Seymour Eaton. Price of Exercise Manual, 50 cents; price of Book of Forms, $1.00. American Book Company, New York and Chicago.

This manual provides a course of instruction in business which may be used to advantage in schools of all grades where the principles of business are taught. The principles of double entry bookkeeping are taught, but the application of principles to the needs of each particular business are left to be learned in that business. The work is planned to encourage original effort. The exercises are drawn largely from actual transactions. The questions are practical and suggestive. The excellent Book of Forms which accompanies the Exercise Manual will serve to make the teaching of this study both easy and effective. J. M. C.

Spencerian System of Penmanship: Common School Course. No. 10, "Connected Business Forms;" No. 11, "Double Entry Bookkeeping." Price, 8 cents each. American Book Company, New York and Chicago.

These books afford the pupil exercise in penmanship, and also familiarize him in a practical manner with ordinary business forms. J. M. C.

Patriotic Citizenship. By Thomas J. Morgan, LL. D. Price, $1.00. 368 pages. American Book Company, New York, Cincinnate, and Chicago. 1895.

The method of this book is a catechism of about 140 questions with as many concise and comprehensive answers by the author. The text of the answers is followed by brief citations from a wide range of authorities chiefly American. Here is found collected much of the finest literature on the selected topics, so arranged as to explain and enforce the text. The book is designed primarily for the public schools following a course in U. S. history, but it is also a good book for the citizen, reading circle, or family. We think the book lacks some of the helps, in the way of outline, contents or index, showing the relation of selected topics to central theme, etc., which would have secured better adaptation from a teaching point of view. The study of this book will give good results in stimulating patriotism and promoting good citizenship. J. M. C.

Elementary Lessons in Algebra. By Stewart B. Sabin and Charles D. Lowry. Price, 50 cents. 128 pages. New York, Cincinnati, and Chicago: American Book Company.

This little book was prepared to meet the demand for a text-book exactly suited to introduce the study of Algebra into Grammar Schools. The development is inductive, and in arrangement, method, problems and exercises, it is well adapted for its purpose. J. M. C.

Elements of Plane Geometry. By John Macnie, A. M., author of "Theory of Equations." Edited by Emerson E. White, A. M., LL. D., author of "White's Series of Mathematics." Price, 75 cents. 240 pages. 1895. New York, Cincinnati, and Chicago: American Book Company.

In this edition the Plane Geometry is bound separately. We reviewed the Plane and Solid Geometry as bound together in our issue of June, 1895, and a further examination gives us no reason to withdraw the favorable comments made on this book in that notice. J. M. C.

Inductive Studies in English Grammar. By William R. Harper, Ph. D., President of the University of Chicago, and Isaac M. Burgess, A. M., Professor in the University of Chicago. Cloth, 12mo, 96 pages. Price, 40 cents. New York, Cincinnati, and Chicago: American Book Company.

This book presents in a brief compass a systematic course in English Grammar, with special reference to its relation and analogy to other languages. The essential facts of the language are briefly and concisely stated, while the terminology and method of presentation are more closely adapted to that used in Latin Grammars. The pupil's knowledge is tested by requiring him to pick out concrete examples of its application from selections of connected English, instead of giving rules with classified groups of examples. The book is scholarly and has many strong points, and is excellently adapted for a review course in English preparatory to the study of the Ancient or Modern Languages. We believe it will meet with a wider use for this purpose and as supplemental to other grammars than as an independent class-book. J. M. C.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single copies, 10 cents. Irvington-on-the-Hudson, New York.

What is probably the most important discussion of the educational question ever held, has been opened in the April *Cosmopolitan.* President Gilman of the Johns Hopkins

University will follow the introductory article, and the leading educators of the day will contribute articles upon this most important inquiry: "Does Modern Education Educate, in the Broadest and Most Liberal Sense of the Term?" Those interested in the instruction of youth, either as teacher or parent, can not afford to miss this remarkable symposium, intended to review the mistakes of the nineteenth century, and signalize the entrance of the twentieth by advancing the cause of education. President Dwight of Yale, Presi- Schurman of Cornell, Bishop Potter and President Morton are among those who have already agreed to contribute to what promises to be the most significant series of educational papers ever printed. The aim is to consider existing methods in the light of the requirements of the life of to-day, and this work has never been undertaken on a scale in any degree approaching that outlined for *The Cosmopolitan*. Write to us for subscriptions.

B. F. F.

The Arena. An Illustrated Monthly Magazine. Edited by John Clarke Redpath and Helen H. Gardner. Price, $3.00 per year, in advance. Single Number, 25 Cents. Boston: The Arena Co.

The April number of *The Arena* is fully up to the average. In the opening article Governor Pingree, Mayor of Detroit, continues the discussion of Municipal Reform begun in the March number by Mayor Quincy, of Boston. Mayor Pingree, in his breezy paper, affirms that "contracts are the centre and almost the entire circumference of municipal government," and that "almost all the bribes of serious influence in municipalities are given for contracts." His remedy is the letting of contracts by referendum, or direct popular vote.

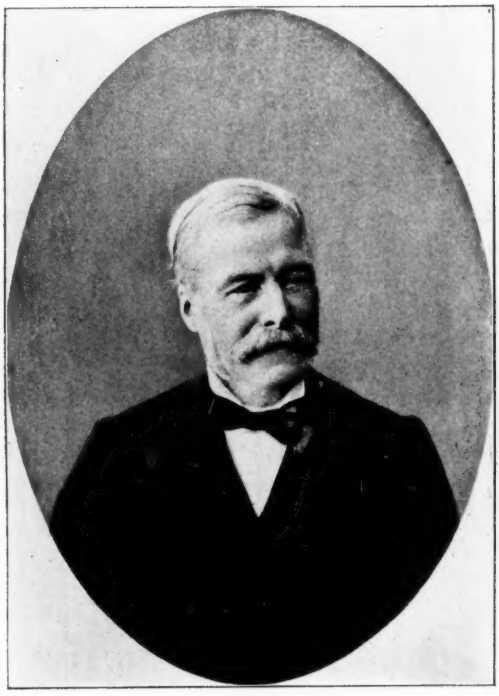
Under the title of "Lincoln and the Matson Negroes," Jesse W. Weik details the history of a curious slave case, the records of which he has recently unearthed, in which Lincoln was concerned, and which was tried in the circuit court in Illinois, in 1847, during the old fugitive-slave day. None of the numerous biographies of Lincoln makes mention of his part in the affair.

B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single Number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York City.

In the "Progress of the World" department of the April *Review of Reviews*, the editor comments on the change of administration at Washington, on the tariff bill, and other measures before the extra session of Congress, and on President McKinley's diplomatic appointments; the Greco-Cretan situation is carefully reviewed, and other recent developments in foreign politics are treated with the thoroughness and impartiality to which the *Review's* readers have grown accustomed.

B. F. F.



GUILLAUME JULES HOÜEL.